S620 - Introduction To Statistical Theory - Homework 5

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1. Suppose that $x_1, \ldots, x_5 \sim Poisson(\lambda)$, i.e., $P(X_i = r) = \dfrac{e^{-\lambda}\lambda^r}{r!}$ for $r = 0, 1, 2, \ldots$. Compare the following two tests of $H_0 : \lambda = 1/2$ versus $H_1 : \lambda = 1$

   (a) Let $y = x_1 + \ldots + x_5$ and reject H_0 iff $y \geq 4$. Note that $Y \sim Poisson(5\lambda)$.

   (b) Reject H_0 iff at least three of the x_i are nonzero.

   Solution: For each test, let us compute its size and power:

   (a) *Size:* Since this is a non-randomized test, take:

$$P_{\lambda=\frac{1}{2}}(\text{reject } H_0) = P_{\lambda=\frac{1}{2}}(y \geq 4) = 1 - P_{\lambda=\frac{1}{2}}(y < 4) = 1 - \sum_{i=0}^{3} P_{\lambda=\frac{1}{2}}(y = i)$$

Since $y \sim Pois(5\lambda) = Pois(\frac{5}{2})$, we have that $P_{\lambda=\frac{1}{2}}(y = i) = \dfrac{e^{-5/2}(\frac{5}{2})^i}{i!}$, and hence:

$$1 - \sum_{i=0}^{3} P_{\lambda=\frac{1}{2}}(y = i) = 1 - \left[e^{-5/2}\left(\frac{(\frac{5}{2})^0}{0!} + \frac{(\frac{5}{2})^1}{1!} + \frac{(\frac{5}{2})^2}{2!} + \frac{(\frac{5}{2})^3}{3!}\right)\right] = 1 - \left[e^{-5/2}\left(1 + \frac{5}{2} + \frac{5^2}{2^3} + \frac{5^3}{2^4 \cdot 3}\right)\right] = 0.242423867$$

And so the size of the test is 0.242423867. Hence, there is approximately a 24.24% chance of rejecting the null given that is true, i.e., committing a type I error.

Power: we would need to do the same computation under the alternative hypothesis:

$$P_{\lambda=1}(\text{reject } H_0) = P_{\lambda=1}(y \geq 4) = 1 - P_{\lambda=1}(y < 4) = 1 - \sum_{i=0}^{3} P_{\lambda=1}(y = i)$$

Since $y \sim Pois(5\lambda) = Pois(5)$, we have that $P_{\lambda=1}(y = i) = \dfrac{e^{-5}5^i}{i!}$, and hence:

$$1 - \sum_{i=0}^{3} P_{\lambda=1}(y = i) = 1 - \left[e^{-5}\left(\frac{5^0}{0!} + \frac{5^1}{1!} + \frac{5^2}{2!} + \frac{5^3}{3!}\right)\right] = 1 - \left[e^{-5}\left(1 + 5 + \frac{5^2}{2} + \frac{5^3}{3 \cdot 2}\right)\right] = 0.734974085$$

This test has high power. In other words, there is approximately a 73.49% chance of rejecting the null hypothesis when the alternative is the true state of nature.

   (b) For this test, let us define for $i = 1, 2, 3, 4, 5$:

$$z_i = \begin{cases} 1 & \text{if } x_i > 0 \\ 0 & \text{otherwise} \end{cases}$$

Now, $Pr_\theta\{z_i = 1\} = Pr_\theta\{x_i > 0\} = 1 - Pr_\theta\{x_i = 0\} = 1 - \dfrac{e^{-\theta}(\frac{5}{2})^0}{0!} = 1 - e^{-\theta}$.
Thus, $z_i \sim Bernoulli(1 - e^{-\theta})$. Define $T_\theta = z_1 + z_2 + \cdots + z_5$. Then, $T_\theta \sim Bin(5, 1 - e^{-\theta})$.

Size: Since this is a non-randomized test, take:

$$\begin{aligned} Pr_{\lambda=\frac{1}{2}}(\text{reject } H_0) &= P(T_{\frac{1}{2}} \geq 3) \\ &= 1 - \left[P(T_{\frac{1}{2}} = 0) + P(T_{\frac{1}{2}} = 1) + P(T_{\frac{1}{2}} = 2)\right] \\ &= 1 - \left[\tbinom{5}{0}(1 - e^{-1/2})^0(e^{-1/2})^5 + \tbinom{5}{1}(1 - e^{-1/2})^1(e^{-1/2})^4 + \tbinom{5}{2}(1 - e^{-1/2})^2(e^{-1/2})^3\right] \\ &= 1 - \left[(e^{-1/2})^5 + 5(1 - e^{-1/2})(e^{-1/2})^4 + 10(1 - e^{-1/2})^2(e^{-1/2})^3\right] \\ &= 0.306217655 \end{aligned}$$

And so the size of the test is 0.306217655. Hence, there is approximately a 30.621% chance of rejecting the null given that is true, i.e., committing a type I error.

Power: we would need to do the same computation under the alternative hypothesis:

$$
\begin{aligned}
Pr_{\lambda=1}(\text{reject } H_0) = P(T_1 \geq 3) &= 1 - [P(T_1 = 0) + P(T_1 = 1) + P(T_1 = 2)] \\
&= 1 - \left[\binom{5}{0}(1-e^{-1})^0(e^{-1})^5 + \binom{5}{1}(1-e^{-1})^1(e^{-1})^4 + \binom{5}{2}(1-e^{-1})^2(e^{-1})^3\right] \\
&= 1 - \left[(e^{-1})^5 + 5(1-e^{-1})(e^{-1})^4 + 10(1-e^{-1})^2(e^{-1})^3\right] \\
&= 0.736436218
\end{aligned}
$$

This test has high power. In other words, there is approximately a 73.64% chance of rejecting the null hypothesis when the alternative is the true state of nature.

In summary we have:

	Test a	Test b
Size	0.242423867	0.306217655
Power	0.734974085	0.736436218

The test have virtually the same power. However, test (a) has smaller size and thus it is more conservative. In other words, for the same power as test (b) we have that test (a) has a smaller chance of committing type I error. It would make sense to choose test (a) instead of test (b).

3. Suppose that $x_1, \ldots, x_n \sim Normal(\mu, \sigma^2)$, where μ is unknown and $\sigma^2 > 0$ is known. Show that the corresponding family of joint densities of $x = (x_1, \ldots, x_n)$ has monotone likelihood ratio. Given $\alpha \in (0, 1)$ and μ_0, construct a uniformly most powerful test of size α for $H_0 : \mu \leq \mu_0$ versus $H_1 : \mu > \mu_0$.

Solution: By definition, the family of densities $\mathcal{F} = \{f(x;\theta) : \theta \in \Theta \subseteq \mathbb{R}\}$, with real scalar parameter θ is of monotone likelihood ratio (MLR) iff there exists a function $t : \mathcal{X} \to \mathbb{R}$ such that

$$\Lambda(x) = \frac{f(x;\theta_2)}{f(x;\theta_1)}$$

is a nondecreasing function of $t(x)$ for any $\theta_1 \leq \theta_2$.

Let us compute the likelihood ratio. First note that the joint densities of independent, i.i.d normal random variable is given by:

$$
\begin{aligned}
f(x_1, \ldots, x_n; \theta) &= \left[\frac{e^{-\frac{1}{2}\left(\frac{X_1-\theta}{\sigma}\right)^2}}{\sigma\sqrt{2\pi}}\right]\left[\frac{e^{-\frac{1}{2}\left(\frac{X_2-\theta}{\sigma}\right)^2}}{\sigma\sqrt{2\pi}}\right]\cdots\left[\frac{e^{-\frac{1}{2}\left(\frac{X_n-\theta}{\sigma}\right)^2}}{\sigma\sqrt{2\pi}}\right] \\
&= \frac{1}{\sigma^n(2\pi)^{n/2}} e^{\left\{-\frac{1}{2}\left(\frac{X_1-\theta}{\sigma}\right)^2\right\} + \left\{-\frac{1}{2}\left(\frac{X_2-\theta}{\sigma}\right)^2\right\} + \cdots + \left\{-\frac{1}{2}\left(\frac{X_n-\theta}{\sigma}\right)^2\right\}} \\
&= \left[\sigma^n(2\pi)^{n/2}\right]^{-1} e^{-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(x_i-\theta)^2\right]}
\end{aligned}
$$

So the likelihood ratio is (suppose $\theta_1 \leq \theta_2$):

$$\begin{aligned}
\frac{f(x_1,\ldots,x_n;\theta_2)}{f(x_1,\ldots,x_n;\theta_1)} &= \frac{\left[\sigma^n(2\pi)^{n/2}\right]^{-1} e^{-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(x_i-\theta_2)^2\right]}}{\left[\sigma^n(2\pi)^{n/2}\right]^{-1} e^{-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(x_i-\theta_1)^2\right]}} \\
&= \frac{e^{-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(x_i-\theta_2)^2\right]}}{e^{-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(x_i-\theta_1)^2\right]}} \\
&= e^{-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(x_i-\theta_2)^2\right]+\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(x_i-\theta_1)^2\right]} \\
&= e^{\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(x_i-\theta_1)^2-\sum_{i=1}^{n}(x_i-\theta_2)^2\right]} \\
&= e^{\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(x_i-\theta_1)^2-(x_i-\theta_2)^2\right]}
\end{aligned}$$

Note that for any i we have: $(x_i-\theta_1)^2-(x_i-\theta_2)^2 = x_i^2-2x_i\theta_1+\theta_1^2-x_i^2+2x_i\theta_2-\theta_2^2 = \theta_1^2-\theta_2^2+2x_i(\theta_2-\theta_1)$. Hence,

$$\sum_{i=1}^{n}\theta_1^2-\theta_2^2+2x_i(\theta_2-\theta_1) = n(\theta_1^2-\theta_2^2)+2(\theta_2-\theta_1)\sum_{i=1}^{n}x_i\cdot\frac{n}{n} = n\{(\theta_1^2-\theta_2^2)+2(\theta_2-\theta_1)\bar{x}\}$$

And so the likelihood ratio is:

$$\begin{aligned}
\frac{f(x_1,\ldots,x_n;\theta_2)}{f(x_1,\ldots,x_n;\theta_1)} &= e^{\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(x_i-\theta_1)^2-(x_i-\theta_2)^2\right]} \\
&= e^{\left[n\{(\theta_1^2-\theta_2^2)+2(\theta_2-\theta_1)\bar{x}\}/\,2\sigma^2\right]}
\end{aligned}$$

This calculation shows that that $\Lambda(x)$ depends on x through $\bar{x}$. Since $\theta_1 \leq \theta_2 \Longrightarrow \theta_2-\theta_1 \geq 0$, we have that $\Lambda(x)$ is an increasing function of $\bar{x}$. Hence, take $t(x)=\bar{x}$ to conclude that the corresponding family of joint densities of $x=(x_1,\ldots,x_n)$ has monotone likelihood ratio.

Having showed that $x=(x_1,\ldots,x_n)$ has monotone likelihood ratio with respect to $t(x)=\bar{x}$, Theorem 4.2 applies and the test:

$$\phi_0(x) = \begin{cases} 1 & \text{if } t(x)=\bar{x}>t_0 \\ 0 & \text{otherwise} \end{cases}$$

Is UMP among all test of size α. So the test that rejects whenever $\bar{x}>t_0$ is our UMP. In this case we can actually find what t_0 is (for convenience, we will find z_α which corresponds to t_0 after standardization of $\bar{x}$). First note that: $Z=\sqrt{n}(\bar{x}-\mu_0)/\sigma^2 \sim Normal(0,1)$. The probability of rejection under the null is:

$$Pr\{Z>z_\alpha\}=\alpha \Longrightarrow 1-\Phi(z_\alpha)=\alpha \Longrightarrow \Phi(z_\alpha)=1-\alpha \Longrightarrow z_\alpha=\Phi^{-1}(1-\alpha)$$

And so we will reject the null if, after standardizing $\bar{x}$ we observe a value $z_\alpha \geq \Phi^{-1}(1-\alpha)$.

4. Suppose that $x_1, \ldots, x_n \sim Normal(\mu, \sigma^2)$, where μ is known and $\sigma^2 > 0$ is unknown. Given $\alpha \in (0,1)$ and $\sigma_0^2 > 0$, construct a uniformly most powerful test of size α for $H_0 : \sigma^2 \le \sigma_0^2$ versus $H_1 : \sigma^2 > \sigma_0^2$.

Solution: Let us compute the likelihood ratio. We already computed the joint density to be:

$$f(x_1, \ldots, x_n; \mu, \sigma) = \left[\sigma^n (2\pi)^{n/2}\right]^{-1} e^{-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(x_i - \mu)^2\right]}$$

So the likelihood ratio is:

$$\begin{aligned}
\frac{f(x_1, \ldots, x_n; \mu, \sigma_2)}{f(x_1, \ldots, x_n; \mu, \sigma_1)} &= \frac{\left[\sigma_2^n (2\pi)^{n/2}\right]^{-1} e^{-\frac{1}{2\sigma_2^2}\left[\sum_{i=1}^{n}(x_i - \mu)^2\right]}}{\left[\sigma_1^n (2\pi)^{n/2}\right]^{-1} e^{-\frac{1}{2\sigma_1^2}\left[\sum_{i=1}^{n}(x_i - \mu)^2\right]}} \\
&= \left(\frac{\sigma_1}{\sigma_2}\right)^n e^{\left\{-\frac{1}{2\sigma_2^2}\sum_{i=1}^{n}(x_i - \mu)^2 + \frac{1}{2\sigma_1^2}\sum_{i=1}^{n}(x_i - \mu)^2\right\}} \\
&= \left(\frac{\sigma_1}{\sigma_2}\right)^n e^{\left\{\sum_{i=1}^{n}(x_i - \mu)^2 \left(\frac{1}{2\sigma_1^2} - \frac{1}{2\sigma_2^2}\right)\right\}}
\end{aligned}$$

Since $\sigma_2 > \sigma_1 \implies \frac{1}{2\sigma_2^2} < \frac{1}{2\sigma_1^2}$ and so $\frac{1}{2\sigma_1^2} - \frac{1}{2\sigma_2^2} > 0$. Note that μ is know and so $\Lambda(x)$ depends on x through $\sum_{i=1}^{n}(x_i - \mu)^2$. Take $t(x) = \sum_{i=1}^{n}(x_i - \mu)^2$, and this family has monotone likelihood ratio ($t(x)$ is increasing). Apply theorem 4.2 to conclude that the test:

$$\phi_0(x) = \begin{cases} 1 & \text{if } t(x) = \sum_{i=1}^{n}(x_i - \mu)^2 > t_0 \\ 0 & \text{otherwise} \end{cases}$$

Is UMP among all test of size α.